DERIVED CATEGORY OF V_{12} FANO THREEFOLDS

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1. Introduction

A V_{12} Fano threefold is a smooth Fano threefold X of index 1 with $\mathsf{Pic}\, X = \mathbb{Z}$ and $(-K_X)^3 = 12$, see [Is, IP]. Let X be a V_{12} threefold. It was shown by Mukai [Mu] that X admits an embedding into a connected component $\mathsf{LGr}_+(V)$ of the Lagrangian Grassmannian $\mathsf{LGr}(V)$ of Lagrangian (5-dimensional) subspaces in a vector space $V = \mathbb{C}^{10}$ with respect to a nondegenerate quadratic form Q, and moreover, $X = \mathsf{LGr}_+(V) \cap \mathbb{P}^8$. Let $\mathcal{O}_X$ be the structure sheaf and let $\mathcal{U}_+$ denote the restriction to X of the tautological (5-dimensional) subbundle from $\mathsf{LGr}_+(V) \subset \mathsf{Gr}(5, V)$. Then it is easy to show that $(\mathcal{U}_+, \mathcal{O}_X)$ is an exceptional pair in the bounded derived category of coherent sheaves on X, $\mathcal{D}^b(X)$. Therefore, triangulated subcategory $\langle \mathcal{U}_+, \mathcal{O}_X \rangle$ generated by the pair is admissible and there exists a semiorthogonal decomposition $\mathcal{D}^b(X) = \langle \mathcal{U}_+, \mathcal{O}_X, \mathcal{A}_X \rangle$, where $\mathcal{A}_X = {}^\perp\langle \mathcal{U}_+, \mathcal{O}_X \rangle \subset \mathcal{D}^b(X)$ is the orthogonal subcategory. The main result of this note is an equivalence $\mathcal{A}_X \cong \mathcal{D}^b(C^\vee)$, where $C^\vee$ is a curve of genus 7.

The curve $C^\vee$ arising in this way in fact is nothing but the orthogonal section of the Lagrangian Grassmannian considered by Iliev and Markushevich [IM1]. Recall that the components $\mathsf{LGr}_+(V)$ and $\mathsf{LGr}_-(V)$ of the Lagrangian Grassmannian $\mathsf{LGr}(V)$ lie in the dual projective spaces $\mathbb{P}(\mathsf{S}^+\mathsf{V})$ and $\mathbb{P}(\mathsf{S}^-\mathsf{V})$ respectively, where $\mathsf{S}^\pm\mathsf{V}$ are the spinor (16-dimensional) representations of the corresponding spinor group $\mathsf{Spin}(V)$. So, with any linear subspace $\mathbb{P}^8 \subset \mathbb{P}(\mathsf{S}^+\mathsf{V})$ one can associate its orthogonal $(\mathbb{P}^8)^\perp = \mathbb{P}^6 \subset \mathbb{P}(\mathsf{S}^-\mathsf{V})$ and consider following [IM1] the orthogonal section $C^\vee := \mathsf{LGr}_-(V) \cap \mathbb{P}^6$, which can be shown to be a smooth genus 7 curve, whenever X is smooth.

Further, Iliev and Markushevich explained in [IM1] the intrinsic meaning of the curve $C^\vee$ associated to the threefold X. They have shown that it is isomorphic to the moduli space of stable rank 2 vector bundles on X with $c_1 = 1$, $c_2 = 5$. Considering a universal bundle $\mathcal{E}_1$ on $X \times C^\vee$ we obtain the corresponding kernel functor $\Phi_{\mathcal{E}_1} : \mathcal{D}^b(C^\vee) \to \mathcal{D}^b(X)$. It follows from [BO] and [IM1] that $\Phi_{\mathcal{E}_1}$ is fully faithful. Moreover, it can be shown that its image is contained in the orthogonal subcategory $\mathcal{A}_X = {}^\perp\langle \mathcal{U}_+, \mathcal{O}_X \rangle \subset \mathcal{D}^b(X)$. Thus, it remains to check that $\Phi_{\mathcal{E}_1} : \mathcal{D}^b(C^\vee) \to \mathcal{A}_X$ is essentially surjective.

To prove the surjectivity of the functor $\Phi_{\mathcal{E}_1}$ we use the following approach. Take arbitrary smooth hyperplane section $X \supset S := X \cap \mathbb{P}^7 = \mathsf{LGr}_+(V) \cap \mathbb{P}^7$ and consider the orthogonal section $S^\vee = \mathsf{LGr}_-(V) \cap (\mathbb{P}^7)^\perp$. Then both S and $S^\vee$ are $K3$ surfaces, moreover $S^\vee$ is smooth and $C^\vee$ is a hyperplane section of $S^\vee$. Iliev and Markushevich have shown in [IM1] that the moduli space of stable rank 2 vector bundles on S with $c_1 = 1$, $c_2 = 5$ is isomorphic to $S^\vee$, so we can again consider a universal bundle $\mathcal{E}_2$ on $S \times S^\vee$ and the corresponding kernel functor $\Phi_{\mathcal{E}_2} : \mathcal{D}^b(S^\vee) \to \mathcal{D}^b(S)$. Again it follows from [BO] and [IM1] that $\Phi_{\mathcal{E}_2}$ is fully faithful, hence an equivalence by [Br]. Further, it is clear that we have an isomorphism $\mathcal{E}_{1|S \times C^\vee} \cong \mathcal{E}_{2|S \times C^\vee}$, hence the composition of $\Phi_{\mathcal{E}_1}$ with pushforward from $C^\vee$ to $S^\vee$ coincides with the composition of $\Phi_{\mathcal{E}_2}$ with restriction from X to S: $\alpha^* \circ \Phi_{\mathcal{E}_1} \cong \Phi_{\mathcal{E}_2} \circ \beta_*$, where $\alpha : S \to X$ and $\beta : C^\vee \to S^\vee$ are the embeddings. The crucial observation however is that the bundles $\mathcal{E}_1$ on $X \times C^\vee$ and $\mathcal{E}_2$ on $S \times S^\vee$ can be glued on $X \times S^\vee$ so that the kernel functor $\mathcal{D}^b(X) \to \mathcal{D}^b(S^\vee)$ corresponding to the glueing vanishes on the subcategory $\mathcal{A}_X$. In other words, we have $(\beta_* \circ \Phi^*_{\mathcal{E}_1})_{|\mathcal{A}_X} \cong (\Phi^*_{\mathcal{E}_2} \circ \alpha^*)_{|\mathcal{A}_X}$, where $\Phi^*_{\mathcal{E}_1}$ and $\Phi^*_{\mathcal{E}_2}$

are the left adjoint functors. Now the proof goes as follows. Take an object $F \in \mathcal{A}_X$, orthogonal to the image of $\Phi_{\mathcal{E}_1}$. Then $\Phi^*_{\mathcal{E}_1}(F) = 0$. Hence $(\Phi^*_{\mathcal{E}_2} \circ \alpha^*)(F) = (\beta_* \circ \Phi^*_{\mathcal{E}_1})(F) = 0$. But $\Phi_{\mathcal{E}_2}$ is an equivalence, hence $\Phi^*_{\mathcal{E}_2}$ is an equivalence, hence $\alpha^*(F) = 0$. Since these arguments apply to *any* smooth hyperplane section $S \subset X$, it follows that the restriction of such F to any smooth hyperplane section is zero, but this immediately implies that $F = 0$.

Having in mind the semiorthogonal decomposition $\mathcal{D}^b(X) = \langle \mathcal{U}_+, \mathcal{O}_X, \mathcal{D}^b(C^\vee) \rangle$ one can informally say that the nontrivial part of the derived category $\mathcal{D}^b(X)$ is described by the curve $C^\vee$. Therefore, the curve $C^\vee$ should appear in all geometrical questions related to X. As a demonstration of this phenomenon we show that the Fano surface of conics on X is isomorphic to the symmetric square of $C^\vee$. This fact was known to Iliev and Markushevich, see [IM2], however we decided to include our proof into the paper for two reasons: it demonstrates very well how the above semiorthogonal decomposition can be used, and, moreover, the same approach allows to investigate any other moduli space on X.

The paper is organised as follows. In section 2 we recall briefly results of [IM1]. In section 3 we give an explicit description of universal bundles on $X \times C^\vee$ and $S \times S^\vee$ and of their glueing on $X \times S^\vee$. In section 4 we give necessary cohomological computations. In section 5 we consider the derived categories and prove the equivalence $\mathcal{A}_X \cong \mathcal{D}^b(C^\vee)$. Finally, in section 6 we investigate conics on X and prove that the Fano surface F_X is isomorphic to $S^2C^\vee$.

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2. Preliminaries

Fix a vector space $V = \mathbb{C}^{10}$ and a quadratic nondegenerate form Q on V. Let $\mathsf{S}^+\mathsf{V}$, $\mathsf{S}^-\mathsf{V}$ denote the spinor (16-dimensional) representations of the spinor group $\mathsf{Spin}(Q)$. Recall that the spaces $\mathsf{S}^\pm\mathsf{V}$ coincide with the (duals of the) spaces of global sections of the ample generators of the Picard group of connected components $\mathsf{LGr}_\pm(V)$ of the Lagrangian Grassmanian of V with respect to Q. In particular, we have canonical embeddings $\mathsf{LGr}_\pm(V) \to \mathbb{P}(\mathsf{S}^\pm\mathsf{V})$.

Choose a pair of subspaces $A_8 \subset A_9 \subset \mathsf{S}^+\mathsf{V}$, $\dim A_i = i$, and consider the intersections

$$\begin{aligned} S &= \mathsf{LGr}_+(V) \cap \mathbb{P}(A_8) \subset \mathbb{P}(\mathsf{S}^+\mathsf{V}), \\ X &= \mathsf{LGr}_+(V) \cap \mathbb{P}(A_9) \subset \mathbb{P}(\mathsf{S}^+\mathsf{V}). \end{aligned} \tag{1}$$

It is easy to see that if X is smooth then X is a V_{12} threefold, and if S is smooth then S is a polarized $K3$ surface of degree 12.

Theorem 2.1 ([Mu]). *If X is a V_{12} Fano threefold, and $S \subset X$ is its smooth $K3$ surface section, then there exists a pair of subspaces $A_8 \subset A_9 \subset \mathsf{S}^+\mathsf{V}$, such that S and X are obtained by* (1).

Recall that the spinor representations $\mathsf{S}^-\mathsf{V}$ and $\mathsf{S}^+\mathsf{V}$ are canonically dual to each other, and denote by $B_7 \subset B_8 \subset \mathsf{S}^-\mathsf{V}$ the orthogonal subspaces,

$$B_i = A_{16-i}^\perp \subset \mathsf{S}^+\mathsf{V}^* \cong \mathsf{S}^-\mathsf{V},$$

and consider the dual pair

$$\begin{aligned} C^\vee &= \mathsf{LGr}_-(V) \cap \mathbb{P}(B_7) \subset \mathbb{P}(\mathsf{S}^-\mathsf{V}), \\ S^\vee &= \mathsf{LGr}_-(V) \cap \mathbb{P}(B_8) \subset \mathbb{P}(\mathsf{S}^-\mathsf{V}). \end{aligned} \tag{2}$$

Again, it is easy to see that if $S^\vee$ is smooth then $S^\vee$ is a polarized $K3$ surface of degree 12, and if $C^\vee$ is smooth then $C^\vee$ is a canonically embedded curve of genus 7.

We denote by H_X, L_X, and P_X the classes of a hyperplane section, of a line, and of a point in $H^\bullet(X,\mathbb{Z})$. The same notation is used for varieties S, $S^\vee$ and $C^\vee$. For example, $P_{S^\vee} \in H^4(S^\vee,\mathbb{Z})$ stands for the class of a point on $S^\vee$.

Let $\mathcal{U}_+$, $\mathcal{U}_-$ denote the tautological subbundles on $\mathsf{LGr}_+(V) \subset \mathsf{Gr}(5,V)$, $\mathsf{LGr}_-(V) \subset \mathsf{Gr}(5,V)$ respectively, and by $\mathcal{U}_{+x}$, $\mathcal{U}_{-y}$ their fibers at points $x \in \mathsf{LGr}_+(V)$, $y \in \mathsf{LGr}_-(V)$ respectively.

Recall the relation between the canonical duality of $\mathsf{LGr}_\pm(V)$ and the intersection of subspaces.

Lemma 2.2 ([IM1])**.** *Let $x \in \mathsf{LGr}_+(V) \subset \mathbb{P}(\mathsf{S^-V})$, $y \in \mathsf{LGr}_-(V) \subset \mathbb{P}(\mathsf{S^+V})$ and denote by $\langle -,-\rangle$ the duality pairing on $\mathsf{S^+V}\times\mathsf{S^-V}$. Then*

$$\begin{aligned}\langle x,y\rangle \neq 0 \quad &\Leftrightarrow \qquad \mathcal{U}_{+x}\cap\mathcal{U}_{-y} = 0,\\ \langle x,y\rangle = 0 \quad &\Rightarrow \quad \dim\left(\mathcal{U}_{+x}\cap\mathcal{U}_{-y}\right) \geq 2.\end{aligned}$$

It follows that for any $(x,y)\in X\times C^\vee$ or $(x,y)\in S\times S^\vee$ we have $\dim\left(\mathcal{U}_{+x}\cap\mathcal{U}_{-y}\right)\geq 2$.

Lemma 2.3 ([IM1])**.** *We have the following equivalences:*
(i) $C^\vee$ is smooth $\Leftrightarrow$ X is smooth $\Leftrightarrow$ for all $x\in X$, $y\in C^\vee$ we have $\dim(\mathcal{U}_{+x}\cap\mathcal{U}_{-y})=2$;
(ii) $S^\vee$ is smooth $\Leftrightarrow$ S is smooth $\Leftrightarrow$ for all $x\in S$, $y\in S^\vee$ we have $\dim(\mathcal{U}_{+x}\cap\mathcal{U}_{-y})=2$.

The following theorem reveals the intrinsic meaning of the curve $C^\vee$ and of the surface $S^\vee$ in terms of X and S respectively.

Theorem 2.4 ([IM1])**.** *(i) The curve $C^\vee$ is the fine moduli space of stable rank 2 vector bundles E on X with $\mathsf{c}_1(E)=H_X$, $\mathsf{c}_2(E)=5L_X$. If E_y, $E_{y'}$ are the bundles on X corresponding to points $y,y'\in C^\vee$, then*

$$\mathsf{Ext}^p(E_y,E_{y'}) = \begin{cases}\mathbb{C}, & \text{for } p=0,1 \text{ and } y=y'\\ 0, & \text{otherwise}\end{cases}$$

(ii) The surface $S^\vee$ is the fine moduli space of stable rank 2 vector bundles E on S with $\mathsf{c}_1(E)=H_S$, $\mathsf{c}_2(E)=5P_S$. If E_y, $E_{y'}$ are the bundles on S corresponding to points $y,y'\in S^\vee$, then

$$\mathsf{Ext}^p(E_y,E_{y'}) = \begin{cases}\mathbb{C}, & \text{for } p=0,2 \text{ and } y=y'\\ \mathbb{C}^2, & \text{for } p=1 \text{ and } y=y'\\ 0, & \text{otherwise}\end{cases}$$

3. The universal bundles

Consider one of the following two products

$$\text{either}\quad W_1 = X\times C^\vee, \qquad\qquad \text{or}\quad W_2 = S\times S^\vee.$$

Denote by $\mathcal{U}_+$ and $\mathcal{U}_-$ the pullbacks of the tautological subbundles on $\mathsf{LGr}_+(V)$ and $\mathsf{LGr}_-(V)$ to $W_i\subset\mathsf{LGr}_+(V)\times\mathsf{LGr}_-(V)$, and consider the following natural composition of morphisms of vector bundles on W_i

$$\xi_i : \mathcal{U}_- \to V\otimes\mathcal{O}_{W_i} \overset{Q}{\cong} V^*\otimes\mathcal{O}_{W_i} \to \mathcal{U}_+^*.$$

Lemma 3.1. *If X (resp. S) is smooth then the rank of ξ_1 (resp. ξ_2) equals 3 at every point of W_1 (resp. W_2).*

Proof: Since the kernel of the natural projection $V^*\otimes\mathcal{O}_{\mathsf{LGr}_+(V)}\to\mathcal{U}_+^*$ equals $\mathcal{U}_+$, it suffices to show that for all points $(x,y)\in W_i\subset\mathsf{LGr}_+(V)\times\mathsf{LGr}_-(V)$ we have $\dim(\mathcal{U}_{+x}\cap\mathcal{U}_{-y})=2$ which follows from lemma 2.3. $\square$

Lemma 3.2. *We have* $\operatorname{Ker}\xi_i \cong (\operatorname{Coker}\xi_i)^*$.

Proof: We have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{U}_- \arrow[r] \arrow[d, "\xi_i"] & V\otimes\mathcal{O}_{W_i} \arrow[r] \arrow[d] & \mathcal{U}_-^* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{U}_+^* \arrow[r] & \mathcal{U}_+^* \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Note that the middle vertical arrow is surjective and its kernel is $\mathcal{U}_+$. Hence, the long exact sequence of kernels and cokernels gives $0 \to \operatorname{Ker}\xi_i \to \mathcal{U}_+ \to \mathcal{U}_-^* \to \operatorname{Coker}\xi_i \to 0$. Moreover, it is clear that the map $\mathcal{U}_+ \to \mathcal{U}_-^*$ in this sequenece coincides with the dual map ξ_i^*. It follows immediately that $\operatorname{Ker}\xi_i \cong \operatorname{Ker}\xi_i^*$. On the other hand, it is clear that $\operatorname{Ker}\xi_i^* \cong (\operatorname{Coker}\xi_i)^*$. □

Let $\mathcal{E}_i$ denote the cokernel of ξ_i on W_i. It follows that $\mathcal{E}_i$ is a rank 2 vector bundle on W_i and we have an exact sequence

$$0 \to \mathcal{E}_i^* \to \mathcal{U}_- \xrightarrow{\xi_i} \mathcal{U}_+^* \to \mathcal{E}_i \to 0. \tag{3}$$

Dualizing, we obtain another sequence

$$0 \to \mathcal{E}_i^* \to \mathcal{U}_+ \xrightarrow{\xi_i^*} \mathcal{U}_-^* \to \mathcal{E}_i \to 0. \tag{4}$$

Lemma 3.3. *The Chern classes of bundles* $\mathcal{E}_i$ *are given by the following formulas*

$$\begin{aligned} \mathsf{c}_1(\mathcal{E}_1) &= H_X + H_{C^\vee}, \quad \mathsf{c}_2(\mathcal{E}_1) = \tfrac{7}{12}H_XH_{C^\vee} + 5L_X + \eta, \\ \mathsf{c}_1(\mathcal{E}_2) &= H_S + H_{S^\vee}, \quad \mathsf{c}_2(\mathcal{E}_2) = \tfrac{7}{12}H_SH_{S^\vee} + 5P_S + 5P_{S^\vee}, \end{aligned}$$

with $\eta \in \big(H^3(X,\mathbb{C})\otimes H^1(C^\vee,\mathbb{C})\big)\cap H^4(X\times C^\vee,\mathbb{Z})$.

Proof: It follows from (3) that

$$\mathsf{ch}(\mathcal{U}_+^*) - \mathsf{ch}(\mathcal{U}_-) = \mathsf{ch}(\mathcal{E}_i) - \mathsf{ch}(\mathcal{E}_i^*) = 2\mathsf{ch}_1(\mathcal{E}_i) + 2\mathsf{ch}_3(\mathcal{E}_i).$$

This allows to compute

$$\begin{aligned} \mathsf{ch}_1(\mathcal{E}_1) &= H_X + H_{C^\vee}, \quad \mathsf{ch}_3(\mathcal{E}_1) = -\tfrac{1}{2}P_X, \\ \mathsf{ch}_1(\mathcal{E}_2) &= H_S + H_{S^\vee}, \quad \mathsf{ch}_3(\mathcal{E}_2) = -\tfrac{1}{2}P_S - \tfrac{1}{2}P_{S^\vee}. \end{aligned}$$

Further, it is clear that $\mathsf{c}_1(\mathcal{E}_i) = \mathsf{ch}_1(\mathcal{E}_i)$, and by Künneth formula we have

$$\mathsf{c}_2(\mathcal{E}_1) = a_1H_XH_{C^\vee} + b_1L_X + \eta, \qquad \mathsf{c}_2(\mathcal{E}_2) = a_2H_SH_{S^\vee} + b_2P_S + c_2P_{S^\vee},$$

for some $a_1, b_1, a_2, b_2, c_2 \in \mathbb{Q}$, $\eta \in \big(H^3(X,\mathbb{C})\otimes H^1(C^\vee,\mathbb{C})\big)\cap H^4(X\times C^\vee,\mathbb{Z})$. Further, since the correspondence $S \leftrightarrow S^\vee$ is symmetric, it is clear that $c_2 = b_2$. Finally, a_i and b_i can be found from the equality $3\mathsf{c}_1(\mathcal{E}_i)\mathsf{c}_2(\mathcal{E}_i) = \mathsf{ch}_1(\mathcal{E}_i)^3 - 6\mathsf{ch}_3(\mathcal{E}_i)$. □

Remark 3.4. Using the Riemann–Roch formula on $X\times C^\vee$ one can compute $\eta^2 = 14$.

Corollary 3.5. *The bundle* $\mathcal{E}_1$ *(resp.* $\mathcal{E}_2$*) is a universal family of rank* 2 *vector bundles with* $\mathsf{c}_1 = H_X$, $\mathsf{c}_2 = 5L_X$ *on* X *(resp. with* $\mathsf{c}_1 = H_S$, $\mathsf{c}_2 = 5P_S$ *on* S*).*

Proof: For every $y \in C^\vee$ (resp. $y \in S^\vee$) we denote by $\mathcal{E}_{1y}$ the fiber of $\mathcal{E}_1$ over $X\times y$ and by $\mathcal{E}_{2y}$ the fiber of $\mathcal{E}_2$ over $S\times y$. It will be shown in lemmas 4.3 and 4.5 below that all bundles $\mathcal{E}_{1y}$ on X for $y\in C^\vee$ and all bundles $\mathcal{E}_{2y}$ on S for $y \in S^\vee$ are stable, hence there exist morphisms

$$f_1 : C^\vee \to \mathcal{M}_X(2, H_X, 5L_X), \qquad f_2 : S^\vee \to \mathcal{M}_S(2, H_S, 5P_S),$$

to the moduli spaces of rank 2 vector bundles on X and S with the indicated rank and Chern classes, such that

$$\mathcal{E}_1 = (\mathsf{id}_X \times f_1)^*\mathcal{E}_1' \otimes q_1^*\mathcal{L}_1, \qquad \mathcal{E}_2 = (\mathsf{id}_S \times f_2)^*\mathcal{E}_2' \otimes q_2^*\mathcal{L}_2, \tag{5}$$

where $\mathcal{E}_1'$ and $\mathcal{E}_2'$ are universal families on $X \times \mathcal{M}_X(2, H_X, 5L_X)$ and $S \times \mathcal{M}_S(2, H_S, 5P_S)$ respectively, $q_1 : X \times C^\vee \to C^\vee$ and $q_2 : S \times S^\vee \to S^\vee$ are the projections, and $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles on $C^\vee$ and $S^\vee$ respectively.

It is easy to see that the maps f_1 and f_2 coincide with the maps ρ constructed in [IM1], section 4. Hence they are isomorphisms, and the bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ are universal. □

Let $\alpha : S \to X$ and $\beta : C^\vee \to S^\vee$ denote the embeddings and put $\lambda_1 = \alpha \times \mathsf{id}_{C^\vee}$, $\lambda_2 = \mathsf{id}_S \times \beta$, $\mu_1 = \mathsf{id}_X \times \beta$, $\mu_2 = \alpha \times \mathsf{id}_{S^\vee}$, $\nu = \alpha \times \beta$. Then we have a commutative diagram

$$\begin{tikzcd} & S \times C^\vee \arrow[dl, "\lambda_1"'] \arrow[dd, "\nu"] \arrow[dr, "\lambda_2"] & \\ X \times C^\vee \arrow[dr, "\mu_1"'] & & S \times S^\vee \arrow[dl, "\mu_2"] \\ & X \times S^\vee & \end{tikzcd}$$

Lemma 3.6. *We have canonical isomorphism $\lambda_1^*\mathcal{E}_1 = \lambda_2^*\mathcal{E}_2$.*

Proof: The claim is clear since $\lambda_i^*\mathcal{E}_i$ is the cokernel of $\lambda_i^*\xi_i$, and $\lambda_1^*\xi_1 = \lambda_2^*\xi_2$ by definition of ξ_i. □

We denote the bundle $\lambda_1^*\mathcal{E}_1 = \lambda_2^*\mathcal{E}_2$ on $S \times C^\vee$ by $\mathcal{E}$.

Consider the product $\tilde{W} = X \times S^\vee$ and the composition

$$\tilde{\xi} : \mathcal{U}_i \to V \otimes \mathcal{O}_{\tilde{W}} \cong V^* \otimes \mathcal{O}_{\tilde{W}} \to \mathcal{U}_+^*.$$

It is clear that

$$\mu_i^*\tilde{\xi} = \xi_i. \tag{6}$$

Lemma 3.7. *The rank of $\tilde{\xi}$ equals 5 at $X \times S^\vee \setminus \big(\mu_1(X \times C^\vee) \cup \mu_2(S \times S^\vee)\big)$.*

Proof: Follows from lemma 2.2. □

Let $\tilde{\mathcal{E}}$ denote the cokernel of $\tilde{\xi}$.

Lemma 3.8. *We have exact sequences on $X \times S^\vee$*

$$0 \to \mathcal{U}_- \xrightarrow{\tilde{\xi}} \mathcal{U}_+^* \to \tilde{\mathcal{E}} \to 0,$$

$$0 \to \tilde{\mathcal{E}} \to \mu_{1*}\mathcal{E}_1 \oplus \mu_{2*}\mathcal{E}_2 \to \nu_*\mathcal{E} \to 0.$$

Proof: The first sequence is exact by lemma 3.7 and definition of $\tilde{\mathcal{E}}$. To verify exactness of the second sequence we note that $\mu_i^*\tilde{\mathcal{E}} = \mathcal{E}_i$ by (6) and (3), and the canonical surjective maps $\tilde{\mathcal{E}} \to \mu_{i*}\mu_i^*\tilde{\mathcal{E}} = \mu_{i*}\mathcal{E}_i$ glue to a surjective map $\tilde{\mathcal{E}} \to \mathsf{Ker}(\mu_{1*}\mathcal{E}_1 \oplus \mu_{2*}\mathcal{E}_2 \to \nu_*\mathcal{E})$. On the other hand, it is easy to check that the Chern characters of $\tilde{\mathcal{E}}$ and $\mathsf{Ker}(\mu_{1*}\mathcal{E}_1 \oplus \mu_{2*}\mathcal{E}_2 \to \nu_*\mathcal{E})$ coincide, hence $\tilde{\mathcal{E}} \cong \mathsf{Ker}(\mu_{1*}\mathcal{E}_1 \oplus \mu_{2*}\mathcal{E}_2 \to \nu_*\mathcal{E})$ and we are done. □

Corollary 3.9. *We have exact sequence on $X \times S^\vee$*

$$0 \to \mu_{1*}\mathcal{E}_1 \otimes \mathcal{O}(-H_X) \to \tilde{\mathcal{E}} \to \mu_{2*}\mathcal{E}_2 \to 0.$$

4. Cohomological computations

Lemma 4.1. *The pair $(\mathcal{U}_+, \mathcal{O}_X)$ in $\mathcal{D}^b(X)$ is exceptional. In orther words,*

$$\mathsf{Ext}^k(\mathcal{U}_+,\mathcal{U}_+) = H^k(X,\mathcal{U}_+^* \otimes \mathcal{U}_+) = \mathsf{Ext}^k(\mathcal{O}_X,\mathcal{O}_X) = H^k(X,\mathcal{O}_X) = \begin{cases} \mathbb{C}, & \textit{for } k=0 \\ 0, & \textit{for } k \neq 0 \end{cases}$$

$$H^\bullet(X,\mathcal{U}_+) = 0.$$

Proof: Recall that X is a complete intersection $X = \mathbb{P}(A_9) \cap \mathsf{LGr}_+(V) \subset \mathbb{P}(\mathsf{S}^+\mathsf{V})$ and $\mathsf{S}^+\mathsf{V}/A_9 = B_7^*$. Hence $X \subset \mathsf{LGr}_+(V)$ is the zero locus of a section of the vector bundle $B_7^* \otimes \mathcal{O}_{\mathsf{LGr}_+(V)}(H_{\mathsf{LGr}_+(V)})$. Therefore, the Koszul complex $\Lambda^\bullet(B_7^* \otimes \mathcal{O}_{\mathsf{LGr}_+(V)}(H_{\mathsf{LGr}_+(V)}))$ is a resolution of the structure sheaf $\mathcal{O}_X$ on $\mathsf{LGr}_+(V)$. In other words, we have an exact sequence

$$0 \to \Lambda^7(B_7 \otimes \mathcal{O}_{\mathsf{LGr}_+(V)}(-H_{\mathsf{LGr}_+(V)})) \to \ldots \to \Lambda^1(B_7 \otimes \mathcal{O}_{\mathsf{LGr}_+(V)}(-H_{\mathsf{LGr}_+(V)})) \to \mathcal{O}_{\mathsf{LGr}_+(V)} \to \mathcal{O}_X \to 0.$$

Tensoring it by $\mathcal{U}_+$ and $\mathcal{U}_+^* \otimes \mathcal{U}_+$ we see that it suffices to compute $H^\bullet(\mathsf{LGr}_+(V), F(-kH_{\mathsf{LGr}_+(V)}))$ for $F = \mathcal{O}_{\mathsf{LGr}_+(V)}$, $F = \mathcal{U}_+$ and $F = \mathcal{U}_+^* \otimes \mathcal{U}_+$ and $0 \le k \le 7$. These cohomologies are computed by Borel–Bott–Weil Theorem [D], since all the bundles under the question are the pushforwards of equivariant line bundles on the flag variety of the spinor group $\mathsf{Spin}(V)$. □

Since the canonical class of X equals $-H_X$, the Serre duality on X gives

Corollary 4.2. *We have*

$$H^\bullet(X,\mathcal{U}_+^*(-H_X)) = 0, \qquad H^k(X,\mathcal{U}_+ \otimes \mathcal{U}_+^*(-H_X)) = \begin{cases} \mathbb{C}, & \textit{for } k=3 \\ 0, & \textit{for } k \neq 3 \end{cases}.$$

Lemma 4.3. *For any $y \in C^\vee$ we have $H^p(X,\mathcal{E}_{1y}(-H_X)) = H^p(X,\mathcal{E}_{1y} \otimes \mathcal{U}_+^*(-H_X)) = 0$. In particular, $\mathcal{E}_{1y}$ is stable.*

Proof: Recall that by definition $C^\vee = \mathsf{LGr}_-(V) \cap \mathbb{P}(B_7)$, and $X = \mathsf{LGr}_+(V) \cap \mathbb{P}(A_9)$ with $A_9 = B_7^\perp$. Choose a hyperplane $\mathbb{P}(B_6) \subset \mathbb{P}(B_7)$ such that $\mathbb{P}(B_6)$ intersects $C^\vee$ transversally and doesn't contain y. Take $A_{10} = B_6^\perp$ and consider $\hat{X} = \mathsf{LGr}_+(V) \cap \mathbb{P}(A_{10})$. Then the arguments of lemma 2.3 show that $\hat{X}$ is a smooth Fano fourfold of index 2 containing X as a hyperplane section. Moreover, the arguments similar to that of lemma 3.8 show that the composition of morphisms on $\hat{X}$

$$\hat{\xi} : \mathcal{U}_{-y} \otimes \mathcal{O}_{\hat{X}} \to V \otimes \mathcal{O}_{\hat{X}} \to \mathcal{U}_+^*$$

is injective and its cokernel is isomorphic to the pushforward of $\mathcal{E}_{1y}$ via the embedding $i : X \to \hat{X}$. In other words, we have the following exact sequence on $\hat{X}$:

$$0 \to \mathcal{U}_{-y} \otimes \mathcal{O}_{\hat{X}} \to \mathcal{U}_+^* \to i_*\mathcal{E}_{1y} \to 0, \tag{7}$$

On the other hand, using Borel–Bott–Weil Theorem and the Koszul resolution of $\hat{X} \subset \mathsf{LGr}_+(V)$ along the lines of lemma 4.1 one can compute

$$H^\bullet(\hat{X},\mathcal{U}_+^* \otimes \mathcal{U}_+^*(-H_{\hat{X}})) = H^\bullet(\hat{X},\mathcal{U}_+^*(-H_{\hat{X}})) = H^\bullet(\hat{X},\mathcal{O}_{\hat{X}}(-H_{\hat{X}})) = 0$$

and the claim follows from the cohomology sequences of (7) twisted by $\mathcal{O}_{\hat{X}}(-H_{\hat{X}})$ and $\mathcal{U}_+^*(-H_{\hat{X}})$ respectively, since

$$H^\bullet(X,\mathcal{E}_{1y}(-H_X)) = H^\bullet(\hat{X}, i_*\mathcal{E}_{1y}(-H_{\hat{X}})), \quad H^\bullet(X,\mathcal{E}_{1y} \otimes \mathcal{U}_+^*(-H_X)) = H^\bullet(\hat{X}, i_*\mathcal{E}_{1y} \otimes \mathcal{U}_+^*(-H_{\hat{X}})).$$

□

Lemma 4.4. *For any $y \in C^\vee$ we have $H^1(X,\mathcal{E}_{1y}(-2H_X)) = 0$.*

Proof: Restricting exact sequence (3) to $X = X \times \{y\} \subset X \times C^\vee$, twisting it by $\mathcal{O}_X(-H_X)$ and taking into account lemma 3.3 we obtain exact sequence

$$0 \to \mathcal{E}_{1y}(-2H_X) \to \mathcal{U}_{-y} \otimes \mathcal{O}_X(-H_X) \to \mathcal{U}_+^*(-H_X) \to \mathcal{E}_{1y}(-H_X) \to 0.$$

It follows from corollary 4.2 and lemma 4.3 that $H^1(X, \mathcal{E}_{1y}(-2H_X)) = \mathcal{U}_{-y} \otimes H^1(X, \mathcal{O}_X(-H_X))$, but using Serre duality we have $H^1(X, \mathcal{O}_X(-H_X)) = H^2(X, \mathcal{O}_X)^* = 0$ by lemma 4.1. $\square$

Lemma 4.5. *For any $y \in S^\vee$ we have $H^0(S, \mathcal{E}_{2y}(-H_S)) = 0$. In particular, $\mathcal{E}_{2y}$ is stable.*

Proof: For $y \in C^\vee$ we have $\mathcal{E}_{2y} = \mathcal{E}_{1y|S}$, hence the claim follows from exact sequence

$$H^0(X, \mathcal{E}_{1y}(-H_X)) \to H^0(S, \mathcal{E}_{2y}(-H_S)) \to H^1(X, \mathcal{E}_{1y}(-2H_X)),$$

since the first term vanishes by lemma 4.3, and the third term vanishes by lemma 4.4.

Now we note that while S (and hence $S^\vee$) is fixed we can take for $C^\vee$ any smooth hyperplane section of $S^\vee$, consider the corresponding smooth $X \supset S$, and repeat the above arguments in this situation. Since any point $y \in S^\vee$ lies on a smooth hyperplane section, these arguments prove the claim for all $y \in S^\vee$. $\square$

Corollary 4.6. *For any $y \in C^\vee$ we have $H^\bullet(X, \mathcal{E}_{1y} \otimes \mathcal{U}_+(-H_X)) = 0$.*

Proof: Tensor exact sequence $0 \to \mathcal{U}_+ \to V \otimes \mathcal{O}_X \to \mathcal{U}_+^* \to 0$ with $\mathcal{E}_{1y}(-H_X)$ and consider the cohomology sequence. $\square$

5. Derived categories

Consider the kernel functors taking $\mathcal{E}_1$ and $\mathcal{E}_2$ for kernels:

$$\Phi_1 : \mathcal{D}^b(C^\vee) \to \mathcal{D}^b(X), \qquad \Phi_2 : \mathcal{D}^b(S^\vee) \to \mathcal{D}^b(S), \qquad \Phi_i(-) = Rp_{i*}(Lq_i{}^*(-) \otimes \mathcal{E}_i),$$

where p_i and q_i are the projections onto the first and the second factors:

$$\begin{tikzcd} & X \times C^\vee \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & \\ X & & C^\vee \end{tikzcd} \qquad \begin{tikzcd} & S \times S^\vee \arrow[dl, "p_2"'] \arrow[dr, "q_2"] & \\ S & & S^\vee \end{tikzcd}$$

Theorem 5.1. *The functor Φ_i is fully faithful.*

Proof: According to the result of Bondal and Orlov [BO] it suffices to check that for the structure sheaves of any two points $y_1, y_2 \in C^\vee$ (resp. $y_1, y_2 \in S^\vee$) and all $p \in \mathbb{Z}$ we have

$$\mathsf{Ext}^p(\Phi_i(\mathcal{O}_{y_1}), \Phi_i(\mathcal{O}_{y_2})) = \mathsf{Ext}^p(\mathcal{O}_{y_1}, \mathcal{O}_{y_2}).$$

But clearly $\Phi_i(\mathcal{O}_{y_k}) = \mathcal{E}_{iy_k}$ and it remains to apply corollary 3.5 and theorem 2.4. $\square$

Corollary 5.2. *The functor $\Phi_2 : \mathcal{D}^b(S^\vee) \to \mathcal{D}^b(S)$ is an equivalence.*

Proof: Any fully faithful functor between the derived categories of K3 surfaces is an equivalence, see [Br]. $\square$

Consider the following diagram

$$\begin{tikzcd} X \arrow[r, dotted, no head, "\mathcal{E}_1"] & C^\vee \arrow[d, "\beta"] \\ S \arrow[u, "\alpha"] \arrow[r, dotted, no head, "\mathcal{E}_2"] & S^\vee \end{tikzcd}$$

where the dotted line connecting two varieties means that we consider the corresponding kernel on their product. This diagram induces a diagram of functors

$$\begin{tikzcd} \mathcal{D}^b(X) \arrow[d, "\alpha^*"'] & \mathcal{D}^b(C^\vee) \arrow[l, "\Phi_1"'] \arrow[d, "\beta_*"] \\ \mathcal{D}^b(S) & \mathcal{D}^b(S^\vee) \arrow[l, "\Phi_2"'] \end{tikzcd}$$

which is commutative by lemma 3.6, since the functor $\alpha^* \circ \Phi_1$ is given by the kernel $\lambda_1^*\mathcal{E}_1$, and the functor $\Phi_2 \circ \beta_*$ is given by the kernel $\lambda_2^*\mathcal{E}_2$.

Let $\Phi_1^* : \mathcal{D}^b(X) \to \mathcal{D}^b(C^\vee)$ and $\Phi_2^* : \mathcal{D}^b(S) \to \mathcal{D}^b(S^\vee)$ denote the left adjoint functors. The standard computation shows that these functors are given by the kernels

$$\mathcal{E}_1^*(-H_X)[3] = \mathcal{E}_1(-2H_X - H_{C^\vee})[3] \quad \text{on } X \times C^\vee, \quad \text{and} \quad \mathcal{E}_2^*[2] = \mathcal{E}_2(-H_S - H_{S^\vee})[2] \quad \text{on } S \times S^\vee$$

respectively. Consider the following diagram

$$\begin{tikzcd} \mathcal{D}^b(X) \arrow[r, "\Phi_1^*"] \arrow[d, "\alpha^*"'] & \mathcal{D}^b(C^\vee) \arrow[d, "\beta_*"] \\ \mathcal{D}^b(S) \arrow[r, "\Phi_1^*"] & \mathcal{D}^b(S^\vee) \end{tikzcd}$$

This diagram is no longer commutative, however, the following proposiotion shows that it becomes commutative if one replaces $\mathcal{D}^b(X)$ by its subcategory ${}^\perp\langle \mathcal{U}_+, \mathcal{O}_X\rangle \subset \mathcal{D}^b(X)$.

Proposition 5.3. *The functors $\beta_* \circ \Phi_1^*$ and $\Phi_2^* \circ \alpha^* : \mathcal{D}^b(X) \to \mathcal{D}^b(S^\vee)$ are isomorphic on the subcategory ${}^\perp\langle \mathcal{U}_+, \mathcal{O}_X\rangle \subset \mathcal{D}^b(X)$.*

Proof: It is clear that the functors $\beta_*\circ\Phi_1^*$ and $\Phi_2^*\circ\alpha^*$ are given by the kernels $\mu_{1*}\mathcal{E}_1(-2H_X - H_{S^\vee})[3]$ and $\mu_{2*}\mathcal{E}_2(-H_X - H_{S^\vee})[2]$ on $X \times S^\vee$ respectively. Considering the helix of the exact sequence of corollary 3.9 twisted by $\mathcal{O}(-H_X - H_{S^\vee})$ we see that there exists a distinguished triangle

$$\mu_{2*}\mathcal{E}_2(-H_X - H_{S^\vee})[2] \to \mu_{1*}\mathcal{E}_1(-2H_X - H_{S^\vee})[3] \to \tilde{\mathcal{E}}(-H_X - H_{S^\vee})[3].$$

It remains to show that a kernel functor $\mathcal{D}^b(X) \to \mathcal{D}^b(S^\vee)$ given by the kernel $\tilde{\mathcal{E}}(-H_X - H_{S^\vee})$ vanishes on the triangulated subcategory ${}^\perp\langle \mathcal{U}_+, \mathcal{O}_X\rangle \subset \mathcal{D}^b(X)$.

Note that lemma 3.8 implies that $\tilde{\mathcal{E}}(-H_X - H_{S^\vee})$ is isomorphic to a cone of the morphism $\tilde{\xi} : \mathcal{U}_-(-H_X - H_{S^\vee}) \to \mathcal{U}_+^*(-H_X - H_{S^\vee})$ on $X \times S^\vee$, so it suffices to check that the kernel functors $\mathcal{D}^b(X) \to \mathcal{D}^b(S^\vee)$ given by the kernels $\mathcal{U}_-(-H_X - H_{S^\vee})$ and $\mathcal{U}_+^*(-H_X - H_{S^\vee})$ on $X \times S^\vee$ vanish on the triangulated subcategory ${}^\perp\langle \mathcal{U}_+, \mathcal{O}_X\rangle \subset \mathcal{D}^b(X)$. Let $\tilde{p} : X \times S^\vee \to X$ and $\tilde{q} : X \times S^\vee \to S^\vee$ denote the projections. The straightforward computation using the projection formula and the Serre duality on X shows that for any object $F \in \mathcal{D}^b(X)$ we have

$$\begin{aligned}\Phi_{\mathcal{U}_-(-H_X - H_{S^\vee})}(F) &= R\tilde{q}_*(L\tilde{p}^*(F) \otimes \mathcal{U}_-(-H_X - H_{S^\vee})) = \\ &= \mathsf{R}\Gamma(X, F(-H_X)) \otimes \mathcal{U}_-(-H_{S^\vee}) = \mathsf{RHom}(F, \mathcal{O}_X)^* \otimes \mathcal{U}_-(-H_{S^\vee}),\end{aligned}$$

$$\begin{aligned}\Phi_{\mathcal{U}_+^*(-H_X - H_{S^\vee})}(F) &= R\tilde{q}_*(L\tilde{p}^*(F) \otimes \mathcal{U}_+^*(-H_X - H_{S^\vee})) = \\ &= \mathsf{R}\Gamma(X, F \otimes \mathcal{U}_+^*(-H_X)) \otimes \mathcal{O}_{S^\vee}(-H_{S^\vee}) = \mathsf{RHom}(F, \mathcal{U}_+)^* \otimes \mathcal{O}_{S^\vee}(-H_{S^\vee}).\end{aligned}$$

In particular, the above kernel functors vanish for all objects $F \in {}^\perp\langle \mathcal{U}_+, \mathcal{O}_X\rangle \subset \mathcal{D}^b(X)$ and we are done. ∎

Theorem 5.4. *We have a semiorthogonal decomposition*

$$\mathcal{D}^b(X) = \big\langle \mathcal{U}_+, \mathcal{O}_X, \Phi_1(\mathcal{D}^b(C^\vee))\big\rangle. \tag{8}$$

Proof: It is clear that $\mathcal{O}_X$ is an exceptional bundle, and $\mathcal{U}_+$ is an exceptional bundle by lemma 4.1. Now, let us verify the semiorthogonality. Indeed,

$$\mathsf{Ext}^\bullet(\mathcal{O}_X,\mathcal{U}_+) = H^\bullet(X,\mathcal{U}_+) = 0$$

by lemma 4.1. Moreover, denoting by $\Phi_1^! : \mathcal{D}^b(X) \to \mathcal{D}^b(C^\vee)$ the right adjoint functor and taking either $F = \mathcal{O}_X$, or $F = \mathcal{U}_+$ we see that

$$\mathsf{Ext}^\bullet(\mathcal{O}_y, \Phi_1^!(F)) = \mathsf{Ext}^\bullet(\Phi_1(\mathcal{O}_y), F) = \mathsf{Ext}^\bullet(\mathcal{E}_y, F) = H^p(X, \mathcal{E}_y^* \otimes F) = H^p(X, \mathcal{E}_y \otimes F(-H_X)) = 0$$

by lemma 4.3 and corollary 4.6. Hence $\Phi_1^!(F) = 0$, since $\{\mathcal{O}_y\}_{y\in C^\vee}$ is a spanning class (see [Br]) in $\mathcal{D}^b(C^\vee)$, hence

$$\mathsf{Ext}^\bullet(\Phi_1(G), F) = \mathsf{Ext}^\bullet(G, \Phi_1^!(F)) = \mathsf{Ext}^\bullet(G, 0) = 0$$

for all $G \in \mathcal{D}^b(C^\vee)$.

It remains to check that $\mathcal{D}^b(X)$ is generated by $\mathcal{U}_+$, $\mathcal{O}_X$, and $\Phi_1(\mathcal{D}^b(C^\vee))$ as a triangulated category. Indeed, assume that $F \in {}^\perp\langle\mathcal{U}_+, \mathcal{O}_X, \Phi_1(\mathcal{D}^b(C^\vee))\rangle$. Since $F \in {}^\perp\Phi_1(\mathcal{D}^b(C^\vee))$ we have $\Phi_1^*(F) = 0$. On the other hand, since $F \in {}^\perp\langle\mathcal{U}_+, \mathcal{O}_X\rangle$ we have by proposition 5.3

$$\Phi_2^* \circ \alpha^*(F) = \beta_* \circ \Phi_1^*(F) = 0.$$

But Φ_2 is an equivalence by corollary 5.2, hence Φ_2^* is an equivalence, hence $\alpha^*(F) = 0$.

Now we note, that while X (and hence $C^\vee$) is fixed, we can take for S any smooth hyperplane section of X. Then the above arguments imply that for any $F \in {}^\perp\langle\mathcal{U}_+, \mathcal{O}_X, \Phi_1(\mathcal{D}^b(C^\vee))\rangle \subset \mathcal{D}^b(X)$ its restriction to any smooth hyperplane section is isomorphic to zero. Thus the proof is finished by the following lemma. $\square$

Lemma 5.5. *If X is a smooth algebraic variety and F is a complex of coherent sheaves on X which restriction to every smooth hyperplane section of X is acyclic, then F is acyclic.*

Proof: Assume that F is not acyclic and let k be the maximal integer such that $\mathcal{H}^k(F) \neq 0$. Let $x \in X$ be a point in the support of the sheaf $\mathcal{H}^k(F)$. Choose a smooth hyperplane section $j : S \subset X$ passing through x. Since the restriction functor j^* is right-exact it is clear that $\mathcal{H}^k(Lj^*F) \neq 0$, a contradiction. $\square$

6. Application: the Fano surface of conics

Let F_X denote the Fano surface of conics (rational curves of degree 2) on X.

Lemma 6.1. *If $R \subset X$ is a conic then $\mathcal{U}_{+|R} \cong \mathcal{O}_R \oplus \mathcal{O}_R(-1)^{\oplus 4}$.*

Proof: Since $\mathcal{U}_+$ is a subbundle of the trivial vector bundle $V \otimes \mathcal{O}_X$, and since we have $r(\mathcal{U}_{+|R}) = 5$, $\deg(\mathcal{U}_{+|R}) = -4$ we have $\mathcal{U}_{+|R} \cong \oplus_{j=1}^5 \mathcal{O}_R(-u_j)$, where $u_j \geq 0$ and $\sum u_j = 4$. Thus it suffices to check that $\dim H^0(R,\mathcal{U}_{+|R}) = 1$. Actually, $\dim H^0(R,\mathcal{U}_{+|R}) \geq 1$ follows from above, so it remains to show that $\dim H^0(R,\mathcal{U}_{+|R}) \geq 2$ is impossible.

Indeed, assume $\dim H^0(R,\mathcal{U}_{+|R}) \geq 2$. Choose a 2-dimensional subspace $U \subset H^0(R,\mathcal{U}_{+|R}) \subset H^0(R, V \otimes \mathcal{O}_R) = V$ and consider $V' = U^\perp/U$. Then $\mathsf{LGr}_+(V') \subset \mathsf{LGr}_+(V)$, and it is clear that $R \subset X' := \mathsf{LGr}_+(V') \cap X$. Since X is a plane section of $\mathsf{LGr}_+(V)$, therefore X' is a plane section of $\mathsf{LGr}_+(V')$. But $V' = \mathbb{C}^6$, hence $\mathsf{LGr}_+(V') \cong \mathbb{P}^3$. But a plane section of $\mathbb{P}^3$ containing a conic contains a plane $\mathbb{P}^2$, hence X' contains $\mathbb{P}^2$, hence X contains $\mathbb{P}^2$ which contradicts Lefschetz theorem for X. $\square$

Lemma 6.2. *We have $\bigcap_{y\in C^\vee}\mathcal{U}_{-y}=0$.*

Proof: Assume that $0\neq v\subset\bigcap_{y\in C^\vee}\mathcal{U}_{-y}$ and consider $V''=v^\perp/\mathbb{C}v$. Then $C^\vee\subset\mathsf{LGr}_-(V'')\subset\mathsf{LGr}_-(V)$. Moreover, since $C^\vee$ is a plane section of $\mathsf{LGr}_-(V)$, hence $C^\vee$ is a plane section of $\mathsf{LGr}_-(V'')$. Further, $V''=\mathbb{C}^8$, hence $\mathsf{LGr}_-(V'')$ is a quadric, and a curve which is a plane section of a quadric is a line or a conic. But $C^\vee$ is neither. $\square$

Theorem 6.3. *We have $F_X\cong S^2C^\vee$.*

Proof: Let $R\subset X$ be a conic and consider a decomposition of its structure sheaf with respect to the semiorthogonal decomposition

$$\mathcal{D}^b(X)=\langle\mathcal{O}_X,\mathcal{U}_+^*,\Phi_1(\mathcal{D}^b(C^\vee))\rangle,$$

obtained from the decomposition (8) by mutating $\mathcal{U}_+$ through $\mathcal{O}_X$. To this end we compute

$$\mathsf{Ext}^p(\mathcal{O}_R,\mathcal{O}_X)=H^{3-p}(X,\omega_X\otimes\mathcal{O}_R)^*=H^{3-p}(R,\omega_R)^*=\begin{cases}\mathbb{C}, & \text{if } p=2\\ 0, & \text{otherwise}\end{cases}$$

$$\mathsf{Ext}^p(\mathcal{O}_R,\mathcal{U}_+)=\mathsf{Ext}^{3-p}(\mathcal{U}_+,\omega_X\otimes\mathcal{O}_R)^*=\mathsf{Ext}^{3-p}(\mathcal{U}_{+|R},\omega_R)^*=$$
$$=H^{3-p}(R,\mathcal{U}_+^*\otimes\omega_R)^*=H^{p-2}(R,\mathcal{U}_{+|R})=\begin{cases}\mathbb{C}, & \text{if } p=2\\ 0, & \text{otherwise}\end{cases}$$

by lemma 6.1. Hence the decomposition gives the following exact sequence

$$0\to\mathcal{O}_X\to\mathcal{U}_+^*\to\Phi_1(\Phi_1^!(\mathcal{O}_R))\to\mathcal{O}_R\to 0, \tag{9}$$

where

$$\Phi_1^!:\mathcal{D}^b(X)\to\mathcal{D}^b(C^\vee),\qquad \Phi_1^!(-)=Rq_{1*}(Lp_1^*(-)\otimes\mathcal{E}_1^*(H_{C^\vee}))[1],$$

is the right adjoint to Φ_1 functor.

Lemma 6.4. *$\Phi_1^!(\mathcal{O}_R)$ is a pure sheaf.*

Proof: In order to understand $\Phi_1^!(\mathcal{O}_R)=Rq_{1*}(Lp_1^*(\mathcal{O}_R)\otimes\mathcal{E}_1^*(H_{C^\vee}))[1]\in\mathcal{D}^b(C^\vee)$, we investigate $H^\bullet(X,\mathcal{E}_{1y}^*\otimes\mathcal{O}_R)=H^\bullet(R,\mathcal{E}^*_{1y|R})$ for all $y\in C^\vee$. The sheaf $\mathcal{E}_{1y}^*$ by (3) is a subsheaf of the trivial vector bundle $\mathcal{U}_{-y}\otimes\mathcal{O}_X$, therefore $H^0(R,\mathcal{E}^*_{1y|R})\subset\mathcal{U}_{-y}\subset V$. On the other hand, by (4) we have $\mathcal{E}_{1y}^*=\mathcal{E}_{1y}(-H_X)$, is a subsheaf of $\mathcal{U}_+$, hence $H^0(R,\mathcal{E}^*_{1y|R})\subset H^0(R,\mathcal{U}_{+|R})=\mathbb{C}\subset V$. Therefore, if $H^0(R,\mathcal{E}^*_{1y|R})\neq 0$ for all $y\in C^\vee$ then $\bigcap_{y\in C^\vee}\mathcal{U}_{-y}\neq 0$ which is false by lemma 6.2. Thus for generic $y\in C^\vee$ we have $H^0(R,\mathcal{E}^*_{1y|R})=0$, hence $R^0q_{1*}(Lp_1^*(\mathcal{O}_R)\otimes\mathcal{E}_1^*(H_{C^\vee}))=0$. On the other hand, since R is 1-dimensional we have $R^kq_{1*}(Lp_1^*(\mathcal{O}_R)\otimes\mathcal{E}_1^*(H_{C^\vee}))=0$ for $k\neq 0,1$. Hence $\Phi_1^!(\mathcal{O}_R)$ is a pure sheaf. $\square$

Corollary 6.5. *$\Phi_1^!(\mathcal{O}_R)$ is an artinian sheaf of length 2 on $C^\vee$.*

Proof: Computation of the Chern character of $\Phi_1^!(\mathcal{O}_R)$ via the Grothendieck–Riemann–Roch. $\square$

It follows from above that $\Phi_1^!(\mathcal{O}_R)$ is either the structure sheaf of a length 2 subscheme in $C^\vee$, or $\Phi_1^!(\mathcal{O}_R)=\mathcal{O}_y\oplus\mathcal{O}_y$ for some $y\in C^\vee$. We claim that the second never happens. To this end we need the following

Lemma 6.6. *We have $\Phi_1^*(\mathcal{U}_+^*)=\mathcal{O}_{C^\vee}$.*

Proof: It is clear that

$$\Phi_1^*(\mathcal{U}_+^*) = Rq_{1*}(Lp_1{}^*(\mathcal{U}_+^*) \otimes \mathcal{E}_1^*(-H_X))[3] = Rq_{1*}(\mathcal{E}_1 \otimes \mathcal{U}_+^*(-2H_X - H_{C^\vee}))[3].$$

On the other hand, tensoring (4) with $\mathcal{U}_+^*(-H_X)$ we obtain exact sequence

$$0 \to \mathcal{E}_1 \otimes \mathcal{U}_+^*(-2H_X - H_C^\vee) \to \mathcal{U}_+ \otimes \mathcal{U}_+^*(-H_X) \to \mathcal{U}_-^* \otimes \mathcal{U}_+^*(-H_X) \to \mathcal{E}_1 \otimes \mathcal{U}_+^*(-H_X) \to 0.$$

Lemma 4.3 implies that $R^\bullet q_{1*}(\mathcal{E}_1 \otimes \mathcal{U}_+^*(-H_X)) = 0$ and corollary 4.2 implies that

$$R^\bullet q_{1*}(\mathcal{U}_-^* \otimes \mathcal{U}_+^*(-H_X)) = 0, \qquad R^k q_{1*}(\mathcal{U}_+ \otimes \mathcal{U}_+^*(-H_X)) = \begin{cases} \mathcal{O}_{C^\vee}, & \text{for } k = 3 \\ 0, & \text{for } k \neq 3 \end{cases}$$

and the claim follows from the spectral sequence. □

Lemma 6.7. $\Phi_1^!(\mathcal{O}_R) \neq \mathcal{O}_y \oplus \mathcal{O}_y$.

Proof: If the above would be true then the decomposition (9) would take form

$$0 \to \mathcal{O}_X \to \mathcal{U}_+^* \to \mathcal{E}_{1y} \oplus \mathcal{E}_{1y} \to \mathcal{O}_R \to 0.$$

On the other hand, it follows from lemma 6.6 that

$$\mathsf{Hom}(\mathcal{U}_+^*, \mathcal{E}_{1y}) = \mathsf{Hom}(\mathcal{U}_+^*, \Phi_1(\mathcal{O}_y)) = \mathsf{Hom}(\Phi_1^*(\mathcal{U}_+^*), \mathcal{O}_y) = \mathsf{Hom}(\mathcal{O}_{C^\vee}, \mathcal{O}_y) = \mathbb{C},$$

hence the map $\mathcal{U}_+^* \to \mathcal{E}_{1y} \oplus \mathcal{E}_{1y}$ must have rank 2 and the above sequence is impossible. □

Corollary 6.8. $\Phi_1^!(\mathcal{O}_R)$ *is the structure sheaf of a length* 2 *subscheme in* $C^\vee$*.*

Thus the functor $\Phi_1^!$ induces a map $F_X \to S^2C^\vee$.

Vice versa, if Z is a length 2 subscheme in $C^\vee$ then

$$\mathsf{Hom}(\mathcal{O}_{C^\vee}, \mathcal{O}_Z) = \mathsf{Hom}(\Phi_1^*(\mathcal{U}_+^*), \mathcal{O}_Z) = \mathsf{Hom}(\mathcal{U}_+^*, \Phi_1(\mathcal{O}_Z)).$$

Therefore, the canonical projection $\mathcal{O}_{C^\vee} \to \mathcal{O}_Z$ induces canonical morphism $f : \mathcal{U}_+^* \to \Phi_1(\mathcal{O}_Z)$. Its kernel, being a rank 1 reflexive sheaf with $\mathsf{c}_1 = 0$, must be isomorphic to $\mathcal{O}_X$, and it is easy to show that its cokernel is the structure sheaf of a conic. Therefore, the map $\Phi_1^! : F_X \to S^2C^\vee$ is an isomorphism. □

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